

# FINITE SPEED OF PROPAGATION AND WAITING TIME FOR A THIN FILM MUSKAT PROBLEM

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**ABSTRACT.** Finite speed of propagation is established for non-negative weak solutions to a thin film approximation of the two-phase Muskat problem. The temporal expansion rate of the support matches the scale invariance of the system. Moreover, we determine sufficient conditions on the initial data for the occurrence of waiting time phenomena.

## 1. INTRODUCTION AND MAIN RESULTS

The Muskat problem is a complex free boundary model which was proposed by Muskat [13] to describe the motion of two immiscible fluids with different densities and viscosities in a porous medium with impermeable bottom (such as intrusion of water into oil). In the limit of thin fluid layers it was shown in [7] that the Muskat problem can be approximated by a strongly coupled parabolic system of equations which, when neglecting surface tension effects, reads as follows

$$\begin{cases} \partial_t f &= \partial_x (f \partial_x ((1+R)f + Rg)), \\ \partial_t g &= R_\mu \partial_x (g \partial_x (f + g)), \end{cases} \quad (t, x) \in (0, \infty) \times \mathbb{R}, \quad (1.1a)$$

and is supplemented with initial conditions

$$f(0) = f_0, \quad g(0) = g_0, \quad x \in \mathbb{R}. \quad (1.1b)$$

The constants  $R$  and  $R_\mu$  in (1.1a), which are assumed in this paper to be positive, are defined as

$$R := \frac{\rho_+}{\rho_- - \rho_+} \quad \text{and} \quad R_\mu := \frac{\mu_-}{\mu_+} R,$$

with  $\rho_-$  and  $\mu_-$  [resp.  $\rho_+$  and  $\mu_+$ ] denoting the density and viscosity of the lower fluid [resp. of the upper fluid]. This reduced model retains only the functions  $f = f(t, x) \geq 0$  and  $g = g(t, x) \geq 0$  as unknowns, where  $f$  is the thickness of the lower fluid layer and  $g$  is the thickness of the upper fluid layer, so that  $f + g$  is the total height of the fluids. When  $R_\mu = R$  the system (1.1a) is also a particular case of thin film models derived in [9] in the context of seawater intrusion.

The system (1.1a) is a degenerate parabolic system with a full diffusion matrix and it can be regarded as a two-phase generalization of the porous medium equation. Among salient features of the latter are the finite speed of propagation and waiting time phenomena. Recall that the former means that the support of solutions remains compact if it is initially compact, while a waiting time phenomenon refers to the situation where the solution vanishes at a point of the boundary of the support of its initial condition for some time. Since the system (1.1a) is degenerate and somewhat

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related to the porous medium equation, these two issues are questions which arise naturally and the purpose of this paper is to provide an affirmative answer to both.

There is a huge literature on the finite speed of propagation for degenerate parabolic equations and various methods have been developed to investigate this issue. In particular, for second order parabolic equations, such as the porous medium equation or the  $p$ -Laplacian equation, for which the comparison principle is available, this property can be derived by comparison with suitable subsolutions and supersolutions, see [16] and the references therein. This approach however cannot be extended to higher order equations or to systems, and energy methods have been developed instead, see [2, 3, 4, 15] and the references therein. These methods were applied in particular to the thin film equation which is a fourth order degenerate parabolic equation and also work for second order equations. A few applications to systems of equations can be found in the literature: finite speed of propagation and the occurrence of waiting time phenomena are shown in [5] for the Poisson-Nernst-Planck system which is of diagonal type with lower order coupling and in [8] for the parabolic-elliptic chemotaxis Keller-Segel system which one can view as a nonlocal parabolic equation.

As we shall see below the energy method is sufficiently flexible to be adapted to study the strongly coupled degenerate parabolic system (1.1a). Before stating our result let us introduce the notion of weak solution to (1.1) to be used hereafter. Let  $\mathcal{K}$  denote the positive cone of the Banach space  $L_1(\mathbb{R}, (1+x^2)dx) \cap L_2(\mathbb{R})$  defined by

$$\mathcal{K} := \{u \in L^1(\mathbb{R}, (1+x^2)dx) \cap L^2(\mathbb{R}) : u \geq 0\}, \quad (1.2)$$

and set  $\mathcal{K}^2 := \mathcal{K} \times \mathcal{K}$ .

**Definition 1.1.** *Given  $(f_0, g_0) \in \mathcal{K}^2$ , a pair  $(f, g) : [0, \infty) \rightarrow \mathcal{K}^2$  is a weak solution to (1.1) if*

- (i)  $(f, g) \in L_\infty(0, \infty; L_2(\mathbb{R}; \mathbb{R}^2))$ ,  $(f, g) \in L_2(0, \infty; H^1(\mathbb{R}; \mathbb{R}^2))$ ,
- (ii)  $(f, g) \in C([0, \infty); H^{-1}(\mathbb{R}; \mathbb{R}^2))$  with  $(f, g)(0) = (f_0, g_0)$ ,

and  $(f, g)$  solves the equations (1.1a) in the following sense

$$\begin{cases} \int_{\mathbb{R}} f(t) \xi dx - \int_{\mathbb{R}} f_0 \xi dx + \int_0^t \int_{\mathbb{R}} f(\sigma) [(1+R)\partial_x f + R\partial_x g](\sigma) \partial_x \xi dx d\sigma = 0, \\ \int_{\mathbb{R}} g(t) \xi dx - \int_{\mathbb{R}} g_0 \xi dx + R_\mu \int_0^t \int_{\mathbb{R}} g(\sigma) (\partial_x f + \partial_x g)(\sigma) \partial_x \xi dx d\sigma = 0 \end{cases} \quad (1.3)$$

for all  $\xi \in C_0^\infty(\mathbb{R})$  and  $t \geq 0$ .

The existence of weak solutions to (1.1) is shown in [11] by a variational scheme. The proof relies on the observation that the system (1.1a) is a gradient flow with respect to the 2-Wasserstein metric of the following energy functional

$$\mathcal{E}(f, g) := \frac{1}{2} \int_{\mathbb{R}} [f^2 + R(f+g)^2] dx. \quad (1.4)$$

This approach actually extends to the two dimensional setting as well as to a related fourth order degenerate system which is also a thin film approximation of the Muskat problem additionally incorporating surface tension effects [12]. Let us point out that the uniqueness of solutions to (1.1) is an open problem.

The main results of this paper are the following.

**Theorem 1.2** (Finite speed of propagation). *Let  $(f, g)$  be a weak solution of (1.1). If  $(f, g)$  satisfies the local energy estimate*

$$\begin{aligned} & \int_{\mathbb{R}} [f^2(T) + R(f+g)^2(T)] \zeta^2 dx \\ & + \int_0^T \int_{\mathbb{R}} (f |(1+R)\partial_x f + R\partial_x g|^2 + RR_\mu g |\partial_x f + \partial_x g|^2) \zeta^2 dx dt \\ & \leq \int_{\mathbb{R}} [f^2(0) + R(f+g)^2(0)] \zeta^2 dx \\ & + 4 \int_0^T \int_{\mathbb{R}} [f((1+R)f + Rg)^2 + RR_\mu g (f+g)^2] |\partial_x \zeta|^2 dx dt \end{aligned} \quad (1.5)$$

for all  $\zeta \in W_4^1(\mathbb{R})$  as well as for  $\zeta \equiv 1$ , then  $(f, g)$  has finite speed of propagation. More precisely, if  $a \geq 0, r_0 > 0$ , and  $\text{supp}(f_0 + g_0) \cap (a - r_0, a + r_0) = \emptyset$ , then there exists a positive constant  $C_* = C_*(R, R_\mu)$  such that

$$\text{supp}(f(T) + g(T)) \cap (a - r_0/2, a + r_0/2) = \emptyset \quad \text{for all } T \in (0, C_* r_0^{5/2} / \mathcal{E}^{1/2}(f_0, g_0)].$$

In particular, if  $\text{supp}(f_0 + g_0) \subset [-b_0, b_0]$ , with  $b_0 > 0$ , then there exists a positive constant  $C^* = C^*(R_\mu, R, f_0, g_0)$  such that

$$\text{supp}(f(T) + g(T)) \subset [-b_0 - C^* T^{1/3}, b_0 + C^* T^{1/3}] \quad \text{for all } T > 0.$$

We note that Theorem 1.2 is only valid for weak solutions which satisfy in addition the local energy estimate (1.5). Unfortunately, we are yet unable to derive it for arbitrary weak solutions and it is in particular unclear whether it holds true for the weak solutions we constructed in [11]. We shall show in Section 3 that for each initial data there is at least a weak solution to (1.1) satisfying the local estimate (1.5). To this end we will adapt an approximation scheme from [6] which allows us to obtain a weak solution as a limit of classical solutions to a regularized version of the original system.

Let us also mention that Theorem 1.2 gives no clue concerning the finite speed of propagation for each component taken separately.

*Remark 1.3.* (a) It is shown in [10] that the system (1.1a) has self-similar solutions of the type

$$[(t, x) \mapsto (1+t)^{-1/3}(F, G)((1+t)^{-1/3}x)], \quad \text{for } t \geq 0 \text{ and } x \in \mathbb{R},$$

with compactly supported profiles  $(F, G) \in H^1(\mathbb{R}, \mathbb{R}^2) \cap \mathcal{K}^2$ . Hence, the estimate on the growth rate of the support obtained in Theorem 1.2 matches that of the self-similar solutions and is likely to be optimal.

(b) The constant  $C^*$  in the last statement of Theorem 1.2 only depends on  $f_0$  and  $g_0$  through the energy  $\mathcal{E}(f_0, g_0)$  and the second moments of  $f_0$  and  $g_0$ .

Due to [4], a direct consequence of the local energy estimate (1.5) is the occurrence of waiting time phenomena.

**Theorem 1.4** (Waiting time phenomena). *Let  $(f, g)$  be a weak solution of (1.1) such that (1.5) holds for all  $\zeta \in W_4^1(\mathbb{R})$ . Let  $x_0 \in \overline{\mathbb{R} \setminus \text{supp}(f_0 + g_0)}$  be such that*

$$\limsup_{r \rightarrow 0} \frac{1}{r^5} \int_{x_0-r}^{x_0+r} [f_0^2 + R(f_0 + g_0)^2] dx < \infty.$$

*Then there exists a positive time  $T_*$  such that  $x_0 \in \overline{\mathbb{R} \setminus \text{supp}(f(T) + g(T))}$  for all  $T \in (0, T_*)$ .*

Let us now describe the content of this paper: Section 2 is devoted to the proof of the main results. While Theorem 1.4 is a straightforward consequence of (1.5) and [4, Theorem 1.2], the proof of Theorem 1.2 requires several steps and is inspired from [3] which deals with the thin film equation. It is worth pointing out that fewer estimates are available for the system (1.1) as in [3]. The last section is devoted to the existence of weak solutions to (1.1) satisfying the local energy estimate (1.5).

## 2. FINITE SPEED OF PROPAGATION

Throughout this section,  $(f, g)$  is a weak solution of (1.1) which satisfies the local energy estimate (1.5) and

$$w := [f^2 + R(f + g)^2]^{3/4}. \quad (2.1)$$

The function  $w$  inherits some regularity properties of  $(f, g)$  as shown in the following result.

**Lemma 2.1.** *Given non-negative functions  $u, v \in H^1(\mathbb{R})$ , let*

$$z := (u^2 + Rv^2)^{3/4}.$$

*Then  $z \in H^1(\mathbb{R})$  and*

$$\partial_x z = \frac{3}{2} \frac{u \partial_x u + Rv \partial_x v}{\mathbf{1}_{\{0\}}(z) + (u^2 + Rv^2)^{1/4}},$$

*where  $\mathbf{1}_E$  is the characteristic function of the set  $E$ .*

*Proof.* We choose positive functions  $u_n, v_n \in C^\infty(\mathbb{R}) \cap H^1(\mathbb{R})$ ,  $n \geq 1$ , such that  $u_n \rightarrow u$  and  $v_n \rightarrow v$  in  $H^1(\mathbb{R})$  and set

$$z_n := (u_n^2 + Rv_n^2)^{3/4}.$$

Obviously  $z_n^{4/3} \rightarrow z^{4/3}$  in  $L_1(\mathbb{R})$  and it follows from the Hölder continuity of the function  $[x \mapsto |x|^{3/4}]$  that

$$|z_n - z|^{4/3} \leq |z_n^{4/3} - z^{4/3}| \quad \text{for all } n \in \mathbb{N},$$

hence  $z_n \rightarrow z$  in  $L_{4/3}(\mathbb{R})$ . We next note that the sequence  $(z_n)_n$  is bounded in  $H^1(\mathbb{R})$  so that it has a subsequence which converges weakly in  $H^1(\mathbb{R})$  towards a limit which coincides with  $z$  almost everywhere. Consequently  $z$  belongs to  $H^1(\mathbb{R})$  and the formula for  $\partial_x z$  follows by standard arguments.  $\square$

We now derive from (1.5) a local energy estimate for the function  $w$  defined in (2.1) which is at the heart of our analysis.

**Lemma 2.2.** *The function  $w$  defined in (2.1) satisfies*

$$\int_{\mathbb{R}} w^{4/3}(T) \zeta^2 dx + C_1 \int_0^T \int_{\mathbb{R}} |\partial_x w|^2 \zeta^2 dx dt \leq \int_{\mathbb{R}} w^{4/3}(0) \zeta^2 dx + C_2 \int_0^T \int_{\mathbb{R}} w^2 |\partial_x \zeta|^2 dx dt \quad (2.2)$$

for all  $T > 0$  and all  $\zeta \in W_4^1(\mathbb{R})$ . The constants  $C_1$  and  $C_2$  depend only on  $R$  and  $R_\mu$ .

*Proof.* By Lemma 2.1 the function  $w$  belongs to  $H^1(\mathbb{R})$  and

$$\begin{aligned} |\partial_x w|^2 &= \left| \frac{f[(1+R)\partial_x f + R\partial_x g] + Rg(\partial_x f + \partial_x g)}{\mathbf{1}_{\{0\}}(w) + (f^2 + R(f+g)^2)^{1/4}} \right|^2 \\ &\leq \frac{2f}{\mathbf{1}_{\{0\}}(w) + (f^2 + R(f+g)^2)^{1/2}} f |(1+R)\partial_x f + R\partial_x g|^2 \\ &\quad + \frac{2R^2 g}{\mathbf{1}_{\{0\}}(w) + (f^2 + R(f+g)^2)^{1/2}} g |\partial_x f + \partial_x g|^2 \\ &\leq 2 \max \left\{ 1, \frac{\sqrt{R}}{R_\mu} \right\} [f |(1+R)\partial_x f + R\partial_x g|^2 + RR_\mu g |\partial_x f + \partial_x g|^2]. \end{aligned} \quad (2.3)$$

In addition, since  $w^{4/3} \geq \max \{(1+R)f^2, Rg^2\}$ , there holds

$$f((1+R)f + Rg)^2 + RR_\mu g(f+g)^2 \leq 2 \max \left\{ \frac{R_\mu}{\sqrt{R}}, \sqrt{1+R} \right\} w^2.$$

Combining these two inequalities with (1.5) gives the claim.  $\square$

We next recall that the Gagliardo-Nirenberg inequality [14, Theorem 1] states that there exists a constant  $C > 0$  such that

$$\|v\|_2 \leq C \|\partial_x v\|_2^{1/5} \|v\|_{4/3}^{4/5} + C \|v\|_{4/3} \quad \text{for all } v \in H^1((-1, 1)).$$

Using a scaling argument, we deduce from the inequality above that, for  $r > 0$ ,

$$\|v\|_2 \leq C \|\partial_x v\|_2^{1/5} \|v\|_{4/3}^{4/5} + Cr^{-1/4} \|v\|_{4/3} \quad \text{for all } v \in H^1((-r, r)). \quad (2.4)$$

A consequence of the Gagliardo-Nirenberg inequality (2.4) is the following interpolation inequality in the spirit of [3, Lemma 10.1].

**Lemma 2.3.** *There is  $C_3 > 0$  such that, given  $r > 0$  and  $v \in H^1((-r, r))$ , there holds*

$$\|v\|_2^2 \leq C_3 \|\partial_x v\|_2^{14/11} I_r^{6/11} + C_3 r^{-7/2} I_r^{3/2}, \quad (2.5)$$

where

$$I_r := \int_{-r}^r (r - |x|)_+^2 |v(x)|^{4/3} dx.$$

*Proof.* We pick  $\rho \in (0, r)$  arbitrary and infer from the Hölder inequality that

$$\begin{aligned} \int_{-r}^r |v|^{4/3} dx &= \int_{-\rho}^{\rho} |v|^{4/3} dx + \int_{\{\rho < |x| < r\}} |v|^{4/3} dx \\ &\leq \frac{1}{(r-\rho)^2} \int_{-\rho}^{\rho} (r - |x|)_+^2 |v|^{4/3} dx + 2(r-\rho)^{1/3} \|v\|_2^{4/3}. \end{aligned}$$

We now choose  $\rho \in (0, r)$  such that

$$\frac{1}{(r - \rho)^2} \int_{-\rho}^{\rho} (r - |x|)_+^2 |v|^{4/3} dx = 2(r - \rho)^{1/3} \|v\|_2^{4/3}$$

and we obtain

$$\|v\|_{4/3} \leq 4 \|v\|_2^{6/7} I_r^{3/28}. \quad (2.6)$$

Using (2.4) and (2.6) yields

$$\|v\|_2^2 \leq C \|\partial_x v\|_2^{2/5} (\|v\|_2^2)^{24/35} I_r^{6/35} + C r^{-1/2} (\|v\|_2^2)^{6/7} I_r^{3/14},$$

and thus

$$(\|v\|_2^2)^{11/35} \leq C \|\partial_x v\|_2^{2/5} I_r^{6/35} + C r^{-1/2} (\|v\|_2^2)^{6/35} I_r^{3/14}.$$

By Young's inequality we get

$$\begin{aligned} \|v\|_2^2 &\leq C \|\partial_x v\|_2^{14/11} I_r^{6/11} + C r^{-35/22} (\|v\|_2^2)^{6/11} I_r^{15/22} \\ &\leq C \|\partial_x v\|_2^{14/11} I_r^{6/11} + \frac{1}{2} \|v\|_2^2 + C r^{-7/2} I_r^{3/2}, \end{aligned}$$

and the proof is complete.  $\square$

We now introduce additional notation. For  $r > 0$  and  $T > 0$  we set

$$\begin{aligned} u_k(r, T) &:= \int_0^T \int_{-r}^r |\partial_x w(t, x)|^2 (r - |x|)_+^k dx dt \quad \text{for } k \in \{0, 1, 2\}, \\ I(r, T) &:= \sup_{t \in (0, T)} \int_{-r}^r w^{4/3}(t, x) (r - |x|)_+^2 dx, \end{aligned}$$

where  $w$  is defined in (2.1). We first derive from (2.2) an inequality relating  $I(r, T)$ ,  $u_0(r, T)$ , and  $u_2(r, T)$  under suitable constraints on  $r$  and  $T$ .

**Lemma 2.4.** *Consider  $r_0 > 0$  such that  $\text{supp}(f_0 + g_0) \cap (-r_0, r_0) = \emptyset$ . There are positive constants  $C_4$  and  $C_5$  such that, if  $T_0 > 0$  is such that*

$$C_4 T_0 \left(\frac{r_0}{2}\right)^{-7/2} I^{1/2}(r_0, T_0) \leq \frac{1}{2}, \quad (2.7)$$

then

$$\frac{1}{3} I(r, T) + u_2(r, T) \leq C_5 T^{4/5} u_0^{7/5}(r, T) \quad (2.8)$$

for all  $r_0/2 \leq r \leq r_0$  and  $0 < T \leq T_0$ .

*Proof.* Let  $T \in (0, T_0]$  and  $r \in (0, r_0]$ . Setting  $\zeta(x) := (r - |x|)_+$ ,  $x \in \mathbb{R}$ , we observe that the assumptions on  $f_0 + g_0$  guarantees that  $\zeta^2(x) w^{4/3}(0, x) = 0$  for  $x \in \mathbb{R}$  and we infer from (2.2) that

$$\int_{\mathbb{R}} w^{4/3}(t) \zeta^2 dx + C_1 \int_0^t \int_{\mathbb{R}} |\partial_x w|^2 \zeta^2 dx ds \leq C_2 \int_0^t \int_{-r}^r w^2 dx ds$$

for all  $t \in (0, T)$ . Hence, there exists a constant  $C > 0$  such that

$$I(r, T) + u_2(r, T) \leq C \int_0^T \int_{-r}^r w^2 dx ds.$$

Using (2.5) and the Hölder inequality, it follows that

$$\begin{aligned} I(r, T) + u_2(r, T) &\leq CT r^{-7/2} I^{3/2}(r, T) + CI^{6/11}(r, T) \int_0^T \|\partial_x w(s)\|_{L_2((-r, r))}^{14/11} ds \\ &\leq C_4 T r^{-7/2} I^{3/2}(r, T) + CT^{4/11} I^{6/11}(r, T) u_0^{7/11}(r, T). \end{aligned}$$

Since  $I(r, T)$  is a nondecreasing function in both variables  $r$  and  $T$ , the property (2.7), together with Young's inequality and the above inequality, leads us to

$$I(r, T) + u_2(r, T) \leq \frac{1}{2} I(r, T) + \frac{1}{6} I(r, T) + C_5 T^{4/5} u_0^{7/5}(r, T)$$

for all  $r_0/2 \leq r \leq r_0$  and  $0 < T \leq T_0$ . This completes the proof.  $\square$

After this preparation, we are in a position to prove our main results.

*Proof of Theorem 1.2.* Since (1.1) is invariant with respect to translations, we may assume that  $a = 0$  so that  $\text{supp}(f_0 + g_0) \cap (-r_0, r_0) = \emptyset$ . Then  $w^{4/3}(0, x) = 0$  for  $x \in (-r_0, r_0)$  and  $I(r_0, t) \rightarrow 0$  as  $t \rightarrow 0$ , cf. (2.2). Consequently there is  $T_0 > 0$  such that the condition (2.7) in Lemma 2.4 is satisfied.

Let  $T \in (0, T_0)$ . In view of

$$[x \mapsto \|\partial_x w(\cdot, x)\|_{L_2(0, T)}^2] \in L_1(\mathbb{R}),$$

we have

$$\partial_r u_2(r, T) = 2u_1(r, T), \quad \partial_r u_1(r, T) = u_0(r, T) \quad \text{for a.e. } r \in (0, r_0), \quad (2.9)$$

and the Hölder inequality yields

$$u_1(r, T) \leq u_2^{1/2}(r, T) u_0^{1/2}(r, T).$$

This inequality, together with (2.8) which is valid here thanks to the choice of  $T_0$ , gives

$$u_1(r, T) \leq CT^{2/5} u_0^{6/5}(r, T) = CT^{2/5} (\partial_r u_1(r, T))^{6/5} \quad \text{for a.e. } r \in (r_0/2, r_0).$$

Equivalently

$$u_1^{5/6}(r, T) \leq \frac{1}{6C_6} T^{1/3} \partial_r u_1(r, T) \quad \text{for a.e. } r \in (r_0/2, r_0). \quad (2.10)$$

Taking a smaller value of  $T_0$  if necessary, we further assume that

$$T_0^{1/3} u_1^{1/6}(r_0, T_0) \leq C_6 \frac{r_0}{2}. \quad (2.11)$$

Let  $T \in (0, T_0]$  and assume for contradiction that  $u_1(r_0/2, T) > 0$ . Together with the monotonicity properties of  $u_1$  this implies that  $u_1(r, T) > 0$  for all  $r \in [r_0/2, r_0]$ . Thanks to this positivity property we infer from (2.10) that

$$C_6 T^{-1/3} \leq \partial_r \left( u_1^{1/6} \right) (r, T) \quad \text{for a.e. } r \in (r_0/2, r_0).$$

After integration we end up with

$$C_6 T^{-1/3} (r_0 - r) \leq u_1^{1/6}(r_0, T) - u_1^{1/6}(r, T),$$

or equivalently

$$T^{1/3} u_1^{1/6}(r, T) \leq T^{1/3} u_1^{1/6}(r_0, T) - C_6 (r_0 - r), \quad r \in [r_0/2, r_0]. \quad (2.12)$$

Taking  $r = r_0/2$  in (2.12) gives

$$0 < T_0^{1/3} u_1^{1/6}(r_0, T_0) - C_6 \frac{r_0}{2},$$

and contradicts (2.11). Therefore  $u_1(r_0/2, T) = 0$  and it follows from (2.9) that  $u_0(r_0/2, T) = 0$  for all  $T \in (0, T_0]$ . Recalling (2.8) we find that  $I(r_0/2, T) = 0$  for all  $T \in (0, T_0]$ .

We further note that, in view of Theorem 3.1 (b) and (2.3),

$$\begin{aligned} u_1(r, T) &\leq r \int_0^T \|\partial_x w(t)\|_2^2 dt \leq Cr [\mathcal{E}(f_0, g_0) - \mathcal{E}(f(T), g(T))] \leq Cr \mathcal{E}(f_0, g_0), \\ I(r, T) &\leq r^2 \sup_{t \in [0, T]} \|w(t)\|_{4/3}^{4/3} \leq r^2 \sup_{t \in [0, T]} \mathcal{E}(f(t), g(t)) \leq r^2 \mathcal{E}(f_0, g_0), \end{aligned}$$

so that (2.7) and (2.11) are satisfied provided  $T_0 = C_* r_0^{5/2} / \mathcal{E}^{1/2}(f_0, g_0)$  for a sufficiently small constant  $C_* > 0$  depending only on  $R$  and  $R_\mu$ . This proves the first claim of Theorem 1.2.

Finally, let  $\text{supp}(f_0 + g_0) \subset [-b_0, b_0]$ , with  $b_0 > 0$ , and let  $T > 0$  be fixed. Choosing  $r_0 = (T \mathcal{E}^{1/2}(f_0, g_0) / C_*)^{2/5}$ , we have that, for each  $a \geq b_0 + (T \mathcal{E}^{1/2}(f_0, g_0) / C_*)^{2/5}$  [resp.  $a \leq -b_0 - (T \mathcal{E}^{1/2}(f_0, g_0) / C_*)^{2/5}$ ]

$$\text{supp}(f_0 + g_0) \cap (a - r_0, a + r_0) = \emptyset.$$

We then infer from the first statement of Theorem 1.2 that  $\text{supp}(f(T) + g(T)) \cap (a - r_0/2, a + r_0/2) = \emptyset$ , from which follows that

$$\text{supp}(f(T) + g(T)) \subset \left[ -b_0 - \frac{T^{2/5} \mathcal{E}^{1/5}(f_0, g_0)}{2C_*^{2/5}}, b_0 + \frac{T^{2/5} \mathcal{E}^{1/5}(f_0, g_0)}{2C_*^{2/5}} \right] \quad \text{for all } T > 0. \quad (2.13)$$

Consequently,  $(f(T) + g(T))$  is compactly supported for each  $T \geq 0$  and we set

$$\beta(T) := \max \{ b_0, \sup(\text{supp}(f(T) + g(T))) \}.$$

It then follows that  $\beta(T) \rightarrow \beta(0) = b_0$  as  $T \rightarrow 0$ . Since the problem (1.1a) is autonomous the estimate (2.13) yields

$$\beta(T_2) - \beta(T_1) \leq \frac{(T_2 - T_1)^{2/5} \mathcal{E}^{1/5}(f(T_1), g(T_1))}{2C_*^{2/5}} \quad \text{for all } T_2 > T_1. \quad (2.14)$$

Besides, we know from [10, Theorem 4.1 (iv)] (after rescaling), that

$$\mathcal{E}(f(t), g(t)) \leq (1+t)^{-1/3} \left[ \mathcal{E}(f_0, g_0) + \frac{1}{6} \int_{\mathbb{R}} \left( f_0 + \frac{R}{R_\mu} g_0 \right) x^2 dx \right] \leq Ct^{-1/3} \quad \text{for all } t > 0. \quad (2.15)$$

Combining (2.14) and (2.15) yields

$$\beta(T_2) - \beta(T_1) \leq C(T_2 - T_1)^{2/5} T_1^{-1/15} \quad \text{for all } T_2 > T_1 > 0.$$

We are now in the position to apply [3, Lemma 7.6] to the above functional inequality and conclude that there exists a positive constant  $C_{*0}$  depending only on  $R, R_\mu, f_0$ , and  $g_0$  such that

$$\beta(T) \leq b_0 + C_* T^{1/3} \quad \text{for all } T > 0,$$

which is the expected propagation rate. The estimate for the expansion of the left boundary of the support is derived in a similar way.  $\square$



*Proof of Theorem 1.4.* Invoking (2.2), Theorem 1.4 is a particular case of the more general result [4, Theorem 1.2] which we apply with  $k = 1$ ,  $p = 2$ , and  $q = 4/3$ .  $\square$

### 3. WEAK SOLUTIONS SATISFYING THE LOCAL ENERGY ESTIMATE

As already mentioned in the Introduction, we now check that there exists at least a weak solution to (1.1) satisfying the local energy estimate (1.5).

**Theorem 3.1** (Existence of weak solutions). *Given  $(f_0, g_0) \in \mathcal{K}^2$ , where  $\mathcal{K}^2$  is defined in (1.2), there exists at least a weak solution  $(f, g)$  to (1.1), satisfying the local energy estimate (1.5) as well as the following estimates*

$$(a) \quad \|f(T)\|_1 = \|f_0\|_1, \quad \|g(T)\|_1 = \|g_0\|_1,$$

$$(b) \quad \mathcal{H}(f(T), g(T)) + \frac{R}{1+2R} \int_0^T \int_{\mathbb{R}} [|\partial_x f|^2 + R|\partial_x(f+g)|^2] dx dt \leq \mathcal{H}(f_0, g_0),$$

$$(c) \quad \mathcal{E}(f(T), g(T)) + \frac{1}{2} \int_0^T \int_{\mathbb{R}} [f((1+R)\partial_x f + R\partial_x g)^2 + RR_\mu g(\partial_x f + \partial_x g)^2] dx dt \leq \mathcal{E}(f_0, g_0)$$

for all  $T \in (0, \infty)$ . The energy functional  $\mathcal{E}$  is given by (1.4) and the entropy functional  $\mathcal{H}$  is defined as

$$\mathcal{H}(f, g) := \int_{\mathbb{R}} \left( f \ln f + \frac{R}{R_\mu} g \ln g \right) dx.$$

The remainder of this section is devoted to the proof of Theorem 3.1. We split the proof of Theorem 3.1 in two steps: we first truncate the spatial domain to a finite interval  $(-L, L)$ , for some arbitrary  $L > 0$ , and then introduce a regularized system having global classical solutions.

**3.1. A regularized problem.** To be more precise, given  $L > 0$  and  $\varepsilon \in (0, 1)$  we define the Hilbert space

$$H_B^2 := \{u \in H^2((-L, L)) : \partial_x u(\pm L) = 0\}$$

and we note that the elliptic operator  $(1 - \varepsilon^2 \partial_x^2) : H_B^2 \rightarrow L_2((-L, L))$  is an isomorphism. Setting

$$\mathcal{R}_\varepsilon[u] := (1 - \varepsilon^2 \partial_x^2)^{-1} u \in H_B^2 \quad \text{for } u \in L_2((-L, L)), \quad (3.1)$$

we consider the following regularized problem

$$\begin{cases} \partial_t f_\varepsilon &= (1+R)\partial_x(f_\varepsilon \partial_x f_\varepsilon) + R\partial_x((f_\varepsilon - \varepsilon)\partial_x \mathcal{R}_\varepsilon[g_\varepsilon]), \\ \partial_t g_\varepsilon &= R_\mu \partial_x((g_\varepsilon - \varepsilon)\partial_x \mathcal{R}_\varepsilon[f_\varepsilon]) + R_\mu \partial_x(g_\varepsilon \partial_x g_\varepsilon), \end{cases} \quad (t, x) \in (0, \infty) \times (-L, L), \quad (3.2a)$$

supplemented with homogeneous Neumann boundary conditions

$$\partial_x f_\varepsilon(t, \pm L) = \partial_x g_\varepsilon(t, \pm L) = 0, \quad t \in (0, \infty), \quad (3.2b)$$

and with regularized initial data

$$f_\varepsilon(0) = f_{0\varepsilon} := \mathcal{R}_\varepsilon[f_0 \mathbf{1}_{(-L, L)}] + \varepsilon, \quad g_\varepsilon(0) = g_{0\varepsilon} := \mathcal{R}_\varepsilon[g_0 \mathbf{1}_{(-L, L)}] + \varepsilon. \quad (3.2c)$$

Clearly, the regularized initial data satisfy  $(f_{0\varepsilon}, g_{0\varepsilon}) \in H_B^2 \times H_B^2$  and

$$f_{0\varepsilon} \geq \varepsilon, \quad g_{0\varepsilon} \geq \varepsilon. \quad (3.3)$$

The solvability of problem (3.2) is studied in [6, Theorem 2.1] with the help of the quasilinear parabolic theory developed in [1] and we recall the result now.

**Proposition 3.2.** *The problem (3.2) has a unique non-negative classical solution*

$$f_\varepsilon, g_\varepsilon \in C([0, \infty); H^1((-L, L))) \cap C((0, \infty); H_B^2) \cap C^1((0, \infty); L_2((-L, L))).$$

Moreover, we have

$$f_\varepsilon \geq \varepsilon, \quad g_\varepsilon \geq \varepsilon \quad \text{for all } (t, x) \in (0, \infty) \times (-L, L),$$

and

$$\|f_\varepsilon(t)\|_1 = \|f_{0\varepsilon}\|_1 = \|f_0 \mathbf{1}_{(-L, L)}\|_1 + 2\varepsilon L, \quad \|g_\varepsilon(t)\|_1 = \|g_{0\varepsilon}\|_1 = \|g_0 \mathbf{1}_{(-L, L)}\|_1 + 2\varepsilon L$$

for all  $t \geq 0$ .

The solutions constructed in Proposition 3.2 enjoy additional properties, cf. [6, Lemmas 2.4 & 2.6].

**Lemma 3.3.** *Given  $T \in (0, \infty)$ , it holds*

$$\mathcal{H}(f_\varepsilon(T), g_\varepsilon(T)) + \int_0^T \int_{-L}^L \left( \frac{1}{2} |\partial_x f_\varepsilon|^2 + \frac{R}{1+2R} |\partial_x g_\varepsilon|^2 \right) dx dt \leq \mathcal{H}(f_\varepsilon(0), g_\varepsilon(0)) \quad (3.4)$$

and

$$\begin{aligned} \mathcal{E}_\varepsilon(f_\varepsilon(T), g_\varepsilon(T)) &+ \int_0^T \int_{-L}^L [f_\varepsilon |(1+R)\partial_x f_\varepsilon + R\partial_x G_\varepsilon|^2 + RR_\mu g_\varepsilon |\partial_x (F_\varepsilon + g_\varepsilon)|^2] dx dt \\ &\leq \mathcal{E}_\varepsilon(f_{0\varepsilon}, g_{0\varepsilon}) + \varepsilon C_2 \int_0^T \int_{-L}^L (|\partial_x f_\varepsilon|^2 + |\partial_x g_\varepsilon|^2) dx dt, \end{aligned} \quad (3.5)$$

with

$$\begin{aligned} \mathcal{E}_\varepsilon(f_\varepsilon, g_\varepsilon) &:= \frac{1}{2} \left[ (1+R)\|f_\varepsilon\|_2^2 + R\|g_\varepsilon\|_2^2 + R \int_{-L}^L (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon) dx \right], \\ \mathcal{H}(f_\varepsilon, g_\varepsilon) &:= \int_{-L}^L \left( f_\varepsilon \ln f_\varepsilon + \frac{R}{R_\mu} g_\varepsilon \ln g_\varepsilon \right) dx. \end{aligned}$$

As a consequence of Proposition 3.2 and Lemma 3.3, the following result is proved in [6].

**Proposition 3.4** (Weak solutions on a finite interval). *There exist a sequence  $\varepsilon_k \rightarrow 0$  and a pair  $(f, g)$  satisfying*

- (i)  $f \geq 0, g \geq 0$  in  $(0, \infty) \times (-L, L)$ ,
- (ii)  $f, g \in L_\infty(0, \infty; L_2((-L, L))) \cap L_2(0, \infty; H^1((-L, L)))$ ,
- (iii)  $f_{\varepsilon_k} \rightarrow f, g_{\varepsilon_k} \rightarrow g$  in  $L_2((0, T) \times (-L, L))$ ,

and

$$\int_{-L}^L f(T) \xi dx - \int_{-L}^L f_0 \xi dx = - \int_0^T \int_{-L}^L f ((1+R)\partial_x f + R\partial_x g) \partial_x \xi dx dt, \quad (3.6)$$

$$\int_{-L}^L g(T) \xi dx - \int_{-L}^L g_0 \xi dx = -R_\mu \int_0^T \int_{-L}^L g (\partial_x f + \partial_x g) \partial_x \xi dx dt \quad (3.7)$$

for all  $\xi \in W_4^1((-L, L))$  and all  $T > 0$ . Moreover

- (a)  $\|f(T)\|_1 = \|f_0 \mathbf{1}_{(-L, L)}\|_1, \|g(T)\|_1 = \|g_0 \mathbf{1}_{(-L, L)}\|_1,$
- (b)  $\mathcal{H}(f(T), g(T)) + \int_0^T \int_{-L}^L \left[ \frac{1}{2} |\partial_x f|^2 + \frac{R}{1+2R} |\partial_x g|^2 \right] dx dt \leq \mathcal{H}(f_0, g_0),$
- (c)  $\mathcal{E}(f(T), g(T)) + \int_0^T \int_{-L}^L [f((1+R)\partial_x f + R\partial_x g)^2 + RR_\mu g(\partial_x f + \partial_x g)^2] dx dt \leq \mathcal{E}(f_0, g_0)$

for all  $T \in (0, \infty)$ .

**3.2. A local energy estimate.** We now derive a local version of inequality (c) in Proposition 3.4.

**Lemma 3.5.** *Let  $(f, g)$  be the limit of  $((f_{\varepsilon_k}, g_{\varepsilon_k}))_k$  found in Proposition 3.4. Then*

$$\begin{aligned}
& \int_{-L}^L [f^2(T) + R(f+g)^2(T)] \zeta^2 dx \\
& + \int_0^T \int_{-L}^L (f|(1+R)\partial_x f + R\partial_x g|^2 + RR_\mu g|\partial_x f + \partial_x g|^2) \zeta^2 dx dt \\
& \leq \int_{-L}^L [f^2(0) + R(f+g)^2(0)] \zeta^2 dx \\
& + 4 \int_0^T \int_{-L}^L [f((1+R)f + Rg)^2 + RR_\mu g(f+g)^2] |\partial_x \zeta|^2 dx dt
\end{aligned} \tag{3.8}$$

for all  $T > 0$  and all  $\zeta \in W_4^1((-L, L))$ .

*Proof.* We set

$$\begin{aligned}
F_\varepsilon &:= \mathcal{R}_\varepsilon[f_\varepsilon], & G_\varepsilon &:= \mathcal{R}_\varepsilon[g_\varepsilon], \\
U_\varepsilon &:= \sqrt{f_\varepsilon} \partial_x [(1+R)f_\varepsilon + RG_\varepsilon], & V_\varepsilon &:= \sqrt{g_\varepsilon} \partial_x [F_\varepsilon + g_\varepsilon],
\end{aligned}$$

and prove first the claim (3.8) for  $\zeta \in C_0^\infty((-L, L))$ . We multiply the first equation of (3.2a) by  $((1+R)f_\varepsilon + RG_\varepsilon)\zeta^2$  and integrate over  $(-L, L)$  to obtain

$$\int_{-L}^L \partial_t f_\varepsilon ((1+R)f_\varepsilon + RG_\varepsilon) \zeta^2 dx = - \int_{-L}^L \sqrt{f_\varepsilon} U_\varepsilon \partial_x [(1+R)f_\varepsilon + RG_\varepsilon] \zeta^2 dx + I_{1,\varepsilon} \tag{3.9}$$

with

$$I_{1,\varepsilon} := \varepsilon R \int_{-L}^L \partial_x G_\varepsilon \partial_x [(1+R)f_\varepsilon + RG_\varepsilon] \zeta^2 dx.$$

Similarly, multiplying the second equation of (3.2a) by  $R(F_\varepsilon + g_\varepsilon)\zeta^2$  and integrating over  $(-L, L)$  give

$$R \int_{-L}^L \partial_t g_\varepsilon (F_\varepsilon + g_\varepsilon) \zeta^2 dx = -RR_\mu \int_{-L}^L \sqrt{g_\varepsilon} V_\varepsilon \partial_x [F_\varepsilon + g_\varepsilon] \zeta^2 dx + I_{2,\varepsilon} \tag{3.10}$$

with

$$I_{2,\varepsilon} := \varepsilon RR_\mu \int_{-L}^L \partial_x F_\varepsilon \partial_x [F_\varepsilon + g_\varepsilon] \zeta^2 dx.$$

We now observe that

$$\begin{aligned} & \int_{-L}^L \partial_t f_\varepsilon ((1+R)f_\varepsilon + RG_\varepsilon) \zeta^2 dx + R \int_{-L}^L \partial_t g_\varepsilon (F_\varepsilon + g_\varepsilon) \zeta^2 dx \\ &= \frac{1+R}{2} \frac{d}{dt} \|f_\varepsilon \zeta\|_2^2 + \frac{R}{2} \frac{d}{dt} \|g_\varepsilon \zeta\|_2^2 + RJ_\varepsilon, \end{aligned} \quad (3.11)$$

with

$$\begin{aligned} J_\varepsilon &:= \int_{-L}^L (G_\varepsilon \partial_t f_\varepsilon + F_\varepsilon \partial_t g_\varepsilon) \zeta^2 dx \\ &= \frac{d}{dt} \int_{-L}^L (F_\varepsilon G_\varepsilon + \varepsilon^2 \partial_x F_\varepsilon \partial_x G_\varepsilon) \zeta^2 dx + 2\varepsilon^2 \int_{-L}^L (G_\varepsilon \partial_x \partial_t F_\varepsilon + F_\varepsilon \partial_x \partial_t G_\varepsilon) \zeta \partial_x \zeta dx. \end{aligned}$$

Recalling that  $\zeta \in C_0^\infty((-L, L))$ , we have

$$\begin{aligned} \int_{-L}^L (F_\varepsilon G_\varepsilon + \varepsilon^2 \partial_x F_\varepsilon \partial_x G_\varepsilon) \zeta^2 dx &= \frac{1}{2} \int_{-L}^L (F_\varepsilon G_\varepsilon + \varepsilon^2 \partial_x F_\varepsilon \partial_x G_\varepsilon) \zeta^2 dx \\ &\quad + \frac{1}{2} \int_{-L}^L (F_\varepsilon G_\varepsilon + \varepsilon^2 \partial_x F_\varepsilon \partial_x G_\varepsilon) \zeta^2 dx \\ &= \frac{1}{2} \int_{-L}^L (F_\varepsilon G_\varepsilon - \varepsilon^2 \partial_x^2 F_\varepsilon G_\varepsilon) \zeta^2 dx \\ &\quad + \frac{1}{2} \int_{-L}^L (F_\varepsilon G_\varepsilon - \varepsilon^2 F_\varepsilon \partial_x^2 G_\varepsilon) \zeta^2 dx \\ &\quad - \varepsilon^2 \int_{-L}^L G_\varepsilon \partial_x F_\varepsilon \zeta \partial_x \zeta dx - \varepsilon^2 \int_{-L}^L F_\varepsilon \partial_x G_\varepsilon \zeta \partial_x \zeta dx \\ &= \frac{1}{2} \int_{-L}^L (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon) \zeta^2 dx + \varepsilon^2 \int_{-L}^L F_\varepsilon G_\varepsilon \partial_x (\zeta \partial_x \zeta) dx, \end{aligned}$$

while

$$\begin{aligned} \int_{-L}^L (G_\varepsilon \partial_x \partial_t F_\varepsilon + F_\varepsilon \partial_x \partial_t G_\varepsilon) \zeta \partial_x \zeta dx &= - \int_{-L}^L (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon) \zeta \partial_x \zeta dx \\ &\quad - \int_{-L}^L (G_\varepsilon \partial_t F_\varepsilon + F_\varepsilon \partial_t G_\varepsilon) \partial_x (\zeta \partial_x \zeta) dx \\ &= - \int_{-L}^L (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon) \zeta \partial_x \zeta dx \\ &\quad - \frac{d}{dt} \int_{-L}^L F_\varepsilon G_\varepsilon \partial_x (\zeta \partial_x \zeta) dx. \end{aligned}$$

We end up with the following formula for  $J_\varepsilon$ :

$$\begin{aligned} J_\varepsilon = & \frac{1}{2} \frac{d}{dt} \int_{-L}^L (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon) \zeta^2 dx - \varepsilon^2 \frac{d}{dt} \int_{-L}^L F_\varepsilon G_\varepsilon \partial_x (\zeta \partial_x \zeta) dx \\ & - 2\varepsilon^2 \int_{-L}^L (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon) \zeta \partial_x \zeta dx. \end{aligned}$$

After integration over  $(0, T)$ , it follows from (3.9)-(3.11) and the previous identity that

$$\begin{aligned} & \frac{1+R}{2} \|f_\varepsilon(T)\zeta\|_2^2 - \frac{1+R}{2} \|f_\varepsilon(0)\zeta\|_2^2 + \frac{R}{2} \|g_\varepsilon(T)\zeta\|_2^2 - \frac{R}{2} \|g_\varepsilon(0)\zeta\|_2^2 \\ & + \frac{R}{2} \int_{-L}^L (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon)(T) \zeta^2 dx - \frac{R}{2} \int_{-L}^L (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon)(0) \zeta^2 dx \\ & - R\varepsilon^2 \int_{-L}^L (F_\varepsilon G_\varepsilon)(T) \partial_x (\zeta \partial_x \zeta) dx + R\varepsilon^2 \int_{-L}^L (F_\varepsilon G_\varepsilon)(0) \partial_x (\zeta \partial_x \zeta) dx \\ & - 2R\varepsilon^2 \int_0^T \int_{-L}^L (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon) \zeta \partial_x \zeta dx ds \\ = & - \int_0^T \int_{-L}^L \sqrt{f_\varepsilon} U_\varepsilon \partial_x [((1+R)f_\varepsilon + RG_\varepsilon) \zeta^2] dx ds + \int_0^T I_{1,\varepsilon} ds \\ & - RR_\mu \int_0^T \int_{-L}^L \sqrt{g_\varepsilon} V_\varepsilon \partial_x [(F_\varepsilon + g_\varepsilon) \zeta^2] dx ds + \int_0^T I_{2,\varepsilon} ds. \end{aligned}$$

Using Young's inequality we get

$$\begin{aligned} & \frac{1+R}{2} \|f_\varepsilon(T)\zeta\|_2^2 + \frac{R}{2} \|g_\varepsilon(T)\zeta\|_2^2 + \frac{R}{2} \int_{-L}^L (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon)(T) \zeta^2 dx \\ & + K_\varepsilon(T) + \frac{1}{2} \int_0^T \int_{-L}^L [U_\varepsilon^2 + RR_\mu V_\varepsilon^2] \zeta^2 dx ds \\ \leq & \frac{1+R}{2} \|f_\varepsilon(0)\zeta\|_2^2 + \frac{R}{2} \|g_\varepsilon(0)\zeta\|_2^2 + \frac{R}{2} \int_{-L}^L (F_\varepsilon g_\varepsilon + G_\varepsilon f_\varepsilon)(0) \zeta^2 dx \\ & + 2 \int_0^T \int_{-L}^L [f_\varepsilon |(1+R)f_\varepsilon + RG_\varepsilon|^2 + RR_\mu g_\varepsilon |F_\varepsilon + g_\varepsilon|^2] |\partial_x \zeta|^2 dx ds \end{aligned} \quad (3.12)$$

with

$$\begin{aligned} K_\varepsilon(T) := & -R\varepsilon^2 \int_{-L}^L (F_\varepsilon G_\varepsilon)(T) \partial_x (\zeta \partial_x \zeta) dx + R\varepsilon^2 \int_{-L}^L (F_\varepsilon G_\varepsilon)(0) \partial_x (\zeta \partial_x \zeta) dx \\ & - 2R\varepsilon^2 \int_0^T \int_{-L}^L (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon) \zeta \partial_x \zeta dx ds - \int_0^T (I_{1,\varepsilon} + I_{2,\varepsilon}) ds. \end{aligned}$$

According to [6], the convergences of  $(f_{\varepsilon_k})_k$  and  $(g_{\varepsilon_k})_k$  towards  $f$  and  $g$  actually take place in stronger topologies than stated in Proposition 3.16. In fact, for all  $T > 0$

$$f_{\varepsilon_k} \rightarrow f, \quad F_{\varepsilon_k} \rightarrow f, \quad g_{\varepsilon_k} \rightarrow g, \quad G_{\varepsilon_k} \rightarrow g \quad \text{in } L_2(0, T; C([-L, L])), \quad (3.13)$$

$$f_{\varepsilon_k}(0) \rightarrow f_0, \quad F_{\varepsilon_k}(0) \rightarrow f_0, \quad g_{\varepsilon_k}(0) \rightarrow g_0, \quad G_{\varepsilon_k}(0) \rightarrow g_0 \quad \text{in } L_2((-L, L)), \quad (3.14)$$

$$U_{\varepsilon_k} \rightharpoonup U := \sqrt{f}((1+R)\partial_x f + R\partial_x g) \quad \text{in } L_2((0, T) \times (-L, L)) \quad (3.15)$$

$$V_{\varepsilon_k} \rightharpoonup V := \sqrt{g}(\partial_x f + \partial_x g) \quad \text{in } L_2((0, T) \times (-L, L)). \quad (3.16)$$

Furthermore it follows from [6, Lemmas 2.3 & 2.5] that

$$((f_{\varepsilon_k}, g_{\varepsilon_k}, F_{\varepsilon_k}, G_{\varepsilon_k}))_k \text{ are bounded in } L_\infty(0, T; L_2(-L, L)) \cap L_2(0, T; H^1(-L, L)). \quad (3.17)$$

We also infer from (3.13) that

$$\lim_{k \rightarrow \infty} \{ \|(f_{\varepsilon_k} - f)(T)\|_2 + \|(F_{\varepsilon_k} - f)(T)\|_2 + \|(g_{\varepsilon_k} - g)(T)\|_2 + \|(G_{\varepsilon_k} - g)(T)\|_2 \} = 0 \quad (3.18)$$

for almost all  $T > 0$ . We may then take  $\varepsilon = \varepsilon_k$  in (3.12) and let  $k \rightarrow \infty$  to deduce from (3.13)-(3.18) that, for almost all  $T > 0$ ,

$$\begin{aligned} & \frac{1+R}{2} \|f(T)\zeta\|_2^2 + \frac{R}{2} \|g(T)\zeta\|_2^2 + R \int_{-L}^L (fg)(T)\zeta^2 dx \\ & + \frac{1}{2} \int_0^T \int_{-L}^L [U^2 + RR_\mu V^2] \zeta^2 dx ds \\ & \leq \frac{1+R}{2} \|f_0\zeta\|_2^2 + \frac{R}{2} \|g_0\zeta\|_2^2 + R \int_{-L}^L f_0 g_0 \zeta^2 dx \\ & + 2 \int_0^T \int_{-L}^L [f|(1+R)f + Rg|^2 + RR_\mu g|f+g|^2] |\partial_x \zeta|^2 dx ds \end{aligned} \quad (3.19)$$

provided we establish that

$$\lim_{k \rightarrow \infty} K_{\varepsilon_k}(T) = 0. \quad (3.20)$$

**The term  $K_\varepsilon(T)$ .** We are left with proving (3.20) and actually identifying the behavior of  $K_\varepsilon(T)$  as  $\varepsilon \rightarrow 0$ . Owing to (3.14) and (3.18), it is clear that

$$\lim_{k \rightarrow \infty} R\varepsilon_k^2 \int_{-L}^L (F_{\varepsilon_k} G_{\varepsilon_k})(T) \partial_x (\zeta \partial_x \zeta) dx = \lim_{k \rightarrow \infty} R\varepsilon_k^2 \int_{-L}^L (F_{\varepsilon_k} G_{\varepsilon_k})(0) \partial_x (\zeta \partial_x \zeta) dx = 0 \quad (3.21)$$

for almost all  $T > 0$ . It next readily follows from (3.17) that

$$\lim_{\varepsilon \rightarrow 0} \int_0^T I_{1,\varepsilon} dt = \lim_{\varepsilon \rightarrow 0} \int_0^T I_{2,\varepsilon} dt = 0. \quad (3.22)$$

Finally, since

$$\partial_t f_\varepsilon = \partial_x (\sqrt{f_\varepsilon} U_\varepsilon - R\varepsilon \partial_x G_\varepsilon),$$

the boundedness (3.15) and (3.17) of  $(U_\varepsilon)_\varepsilon$  in  $L_2((0, T) \times (-L, L))$  and  $(f_\varepsilon)_\varepsilon$  in  $L_\infty(0, T; L_2(-L, L))$  implies that  $(\sqrt{f_\varepsilon} U_\varepsilon - R\varepsilon \partial_x G_\varepsilon)_\varepsilon$  is bounded in  $L_2(0, T; L_4(-L, L))$ . Consequently, the sequence  $(\partial_t f_\varepsilon)_\varepsilon$  is bounded in  $L_2(0, T; (W_4^1(-L, L))')$  and so is  $(\partial_t g_\varepsilon)_\varepsilon$  by a similar argument. Owing to the properties of  $(1 - \varepsilon^2 \partial_x^2)^{-1}$  we conclude that

$$(\partial_t F_\varepsilon)_\varepsilon, (\partial_t G_\varepsilon)_\varepsilon \text{ are bounded in } L_2(0, T; (W_4^1)'), \quad (3.23)$$

see [6, Lemma 3.1] for a similar result. Now, since  $\zeta \in C_0^\infty((-L, L))$  and  $W_4^1(-L, L)$  is an algebra, we infer from (3.23) that

$$\begin{aligned} & 2R\varepsilon^2 \left| \int_0^T \int_{-L}^L (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon) \zeta \partial_x \zeta \, dx \, ds \right| \\ & \leq C\varepsilon^2 \int_0^T \left( \|\partial_t F_\varepsilon\|_{(W_4^1)'} \|\partial_x G_\varepsilon\|_{W_4^1} + \|\partial_t G_\varepsilon\|_{(W_4^1)'} \|\partial_x F_\varepsilon\|_{W_4^1} \right) \|\zeta \partial_x \zeta\|_{W_4^1} \, ds \\ & \leq C(\zeta, T) \varepsilon^2 \left[ \int_0^T \left( \|\partial_x G_\varepsilon\|_{W_4^1}^2 + \|\partial_x F_\varepsilon\|_{W_4^1}^2 \right) ds \right]^{1/2}. \end{aligned} \quad (3.24)$$

Now, owing to (3.1), for almost all  $t \in (0, T)$  the function  $\partial_x F_\varepsilon(t)$  solves

$$\partial_x F_\varepsilon - \varepsilon^2 \partial_x^2 \partial_x F_\varepsilon = \partial_x f_\varepsilon \quad \text{in } (-L, L), \quad \partial_x F_\varepsilon(\pm L) = 0,$$

which implies that

$$\|\partial_x F_\varepsilon\|_2^2 + \varepsilon^2 \|\partial_x^2 F_\varepsilon\|_2^2 + \varepsilon^4 \|\partial_x^3 F_\varepsilon\|_2^2 \leq C \|\partial_x f_\varepsilon\|_2^2.$$

These estimates along with the Gagliardo-Nirenberg inequality [14, Theorem 1] give

$$\begin{aligned} \|\partial_x F_\varepsilon\|_{W_4^1} & \leq C (\|\partial_x F_\varepsilon\|_4 + \|\partial_x^2 F_\varepsilon\|_4) \\ & \leq C \left( \|\partial_x^2 F_\varepsilon\|_2^{1/4} \|\partial_x F_\varepsilon\|_2^{3/4} + \|\partial_x^3 F_\varepsilon\|_2^{1/4} \|\partial_x^2 F_\varepsilon\|_2^{3/4} + \|\partial_x^2 F_\varepsilon\|_2 \right) \\ & \leq C \varepsilon^{-5/4} \|\partial_x f_\varepsilon\|_2. \end{aligned}$$

A similar estimate being valid for  $\|\partial_x G_\varepsilon\|_{W_4^1}$  with  $\|\partial_x g_\varepsilon\|_2$  instead of  $\|\partial_x f_\varepsilon\|_2$ , we deduce from (3.17) and (3.24) that

$$\begin{aligned} & 2R\varepsilon^2 \left| \int_0^T \int_{-L}^L (\partial_t F_\varepsilon \partial_x G_\varepsilon + \partial_x F_\varepsilon \partial_t G_\varepsilon) \zeta \partial_x \zeta \, dx \, ds \right| \\ & \leq C(\zeta, T) \varepsilon^{3/4} \left[ \int_0^T (\|\partial_x f_\varepsilon\|_2^2 + \|\partial_x g_\varepsilon\|_2^2) \, ds \right]^{1/2} \leq C(\zeta, T) \varepsilon^{3/4}. \end{aligned} \quad (3.25)$$

Combining (3.21), (3.22), and (3.25) gives the claim (3.20) and completes the proof of (3.8) for  $\zeta \in C_0^\infty(\mathbb{R})$ , its validity for all  $T > 0$  being obtained by a lower semicontinuity argument. According to the regularity of  $(f, g)$  the extension of Lemma 3.5 to all functions  $\zeta \in W_4^1((-L, L))$  follows by a density argument.  $\square$

**3.3. The limit  $L \rightarrow \infty$ .** For each positive  $L$ , we denote the couple found in Proposition 3.4 by  $(f^L, g^L)$ . The family  $((f^L, g^L))_L$  satisfies the same bounds as the family  $((f_\varepsilon, g_\varepsilon))_\varepsilon$ , so that performing the limit  $L \rightarrow \infty$  may be done as the limit  $\varepsilon \rightarrow 0$ , the only difference being the unboundedness of the domain which one has to cope with. To this end we derive the following lemma which controls the behavior at infinity of  $(f^L, g^L)$ .

**Lemma 3.6.** *It holds that*

$$\int_{-L/2}^{L/2} \left( f^L + \frac{R}{R_\mu} g^L \right) (T) x^2 \, dx \leq \int_{-L}^L \left( f_0 + \frac{R}{R_\mu} g_0 \right) x^2 \, dx + T \mathcal{E}(f_0, g_0) \quad (3.26)$$

for all  $T > 0$ .

*Proof.* We define the function

$$\Phi(x) = \begin{cases} -2Lx - x^2 - 3L^2/4 & , \quad -L \leq x \leq -L/2, \\ x^2 & , \quad -L/2 \leq x \leq L/2, \\ 2Lx - x^2 - 3L^2/4 & , \quad L/2 \leq x \leq L. \end{cases}$$

We take  $\xi = \Phi$  in (3.6) and  $\xi = R\Phi/R_\mu$  in (3.7) to obtain that, using integration by parts and the bound  $\Phi'' \leq 2$ ,

$$\begin{aligned} \int_{-L}^L \left( f^L + \frac{R}{R_\mu} g^L \right) (T) \Phi \, dx &= \int_{-L}^L \left( f_0 + \frac{R}{R_\mu} g_0 \right) \Phi \, dx + \frac{1}{2} \int_0^T \int_{-L}^L ((f^L)^2 + R(f^L + g^L)^2) \Phi'' \, dx \, dt \\ &\leq \int_{-L}^L \left( f_0 + \frac{R}{R_\mu} g_0 \right) \Phi \, dx + T\mathcal{E}(f_0, g_0). \end{aligned}$$

In addition

$$x^2 \mathbf{1}_{[-L/2, L/2]} \leq \Phi(x) \leq x^2, \quad \text{for } x \in [-L, L],$$

and the claim follows.  $\square$

Thanks to Lemma 3.6 we may argue as in the proof of Proposition 3.16, see [6], to perform the limit  $L \rightarrow \infty$  and complete the proof of Theorem 3.1. We in particular use Lemma 3.6 to establish the entropy inequality (b) as well as the conservation of mass in Theorem 3.1.

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